Math 1552

Sections 10.8 and 10.9: Taylor Polynomials and Taylor Series

Math 1552 lecture slides adapted from the course materials
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Quiz 4 – Tuesday July 13, 2021
– last 30 minutes of studio

Topics:

- comparison tests (BCT and LCT)
- alternating series
- integral test for convergence
- ratio and root tests for convergence of series
- p-series (know how to recognize and apply)

No poll today.

Learning Goals

- Understand the process to finding a Taylor polynomial for a given function and center
- Estimate a function value using Taylor Polynomials and a specified error range
- Recognize standard formulas for basic MacLaurin series
- Manipulate the standard series to find MacLaurin series for other functions
- Appropriately use error terms for alternating and non-alternating Taylor series

Taylor Polynomial

A *Taylor Polynomial* for a continuous function *f* about *x=a* is defined as:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Note that if *a=0*, the formula reduces to:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k$$

Example 1:

Find the third-degree Taylor polynomial of the function

$$f(x) = \sqrt{x}$$

in powers of (x-1).

→ find $P_N(x)$, when $N=3$ and $a=1$

→ $$P_3(x) = \sum_{k=0}^{3} \frac{f^{(k)}(1)}{k!}(x-1)^k$$

→ compute the derivatives:

$$f^{(0)}(x) = f(x) = \sqrt{x} \longrightarrow f^{(0)}(1) = 1$$

$$f' = f^{(1)}(x) = \frac{1}{2\sqrt{x}} \longrightarrow f^{(1)}(1) = \frac{1}{2}$$

$$f'' = f^{(2)}(x) = -\frac{1}{4x^{3/2}} \longrightarrow f^{(2)}(1) = -\frac{1}{4}$$

$$f^{(3)}(x) = \frac{3}{8x^{5/2}} \longrightarrow f^{(3)}(1) = 3/8$$

$\rightarrow$ so $$P_3(x) = \frac{1}{0!} + \frac{1}{2 \cdot 1!}(x-1) - \frac{1}{4 \cdot 2!}(x-1)^2 + \frac{3}{8 \cdot 3!}(x-1)^3$$

$$= 1 + \frac{(x-1)}{2} - \frac{1}{8}(x-1)^2$$

$$+ \frac{(x-1)^3}{16}$$

<u>Question</u>: *Find a fourth-degree Taylor polynomial for f(x)=cos(x) about x=0.*

e.g., expand about $x=a$

A. $1-\frac{x^2}{2!}+\frac{x^4}{4!}$

B. $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}$

C. $x-\frac{x^3}{3!}$

D. $1+x+x^2+x^3+x^4$

$\rightarrow$ Compute $P_4(x)$ when $a=0$

$$P_4(x)=\sum_{k=0}^{4}\frac{f^{(k)}(0)}{k!}x^k$$

$\rightarrow$ Compute the derivatives:

$$f^{(0)}(x)=f(x)=\cos(x)\longrightarrow f^{(0)}(0)=1$$

$$f^{(1)}(x)=f'(x)=-\sin(x)\longrightarrow f^{(1)}(0)=0$$

$$f^{(2)}(x)=f''(x)=-\cos(x)\longrightarrow f^{(2)}(0)=-1$$

$$f^{(3)}(x) = \sin(x) \longrightarrow f^{(3)}(0) = 0$$

$$f^{(4)}(x) = \cos(x) \longrightarrow f^{(4)}(0) = 1$$

$\longrightarrow$ so $P_4(x) = \displaystyle\sum_{k=0}^{4} \frac{f^{(k)}(0)}{k!} x^k$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

$\longrightarrow$ Note that $\cos(x)$ is even, so we only

get the even powers of x in these polynomial approximations to the function

Taylor Remainder Term

RN

The remainder term for P_n, where c is some number between a and x, is given by:

$c \leq |x-a|$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

We can find an upper bound for the remainder using the formula:

$$|R_n(x)| \leq \max_{c} |f^{(n+1)}(c)| \frac{|x-a|^{n+1}}{(n+1)!}$$

$c \leq |x-a|$

(Over what range of c is the maximum taken?)

Example 2:

Find the maximum error when

$\sqrt{1.5}$ is approximated using a 3rd degree

Taylor polynomial to the function

$f(x)=\sqrt{2-x}.$

$\longrightarrow$ first, let's compute the approximation to f

$\longrightarrow$ Note that $\sqrt{1.5}=\sqrt{\frac{3}{2}}=f(\frac{1}{2})$

$\longrightarrow$ we can expand about $a=0$, and then plug in $x=1/2$ to get an approx. to $\sqrt{1.5}$

→ compute $P_3(x) = \sum_{k=0}^{3} \frac{f^{(k)}(0)}{k!} x^k$

→ find the derivatives!

$f^{(0)}(x) = f(x) = \sqrt{2-x} \longrightarrow f^{(0)}(0) = \sqrt{2}$

$f^{(1)}(x) = f'(x) = -\frac{1}{2\sqrt{2-x}} \longrightarrow f^{(1)}(0) = -\frac{1}{2\sqrt{2}}$

$f^{(2)}(x) = f''(x) = -\frac{1}{4(2-x)^{3/2}} \rightarrow f^{(2)}(0) = -\frac{1}{4 \cdot 2^{3/2}}$

$f^{(3)}(x) = -\frac{3}{8(2-x)^{5/2}} \longrightarrow f^{(3)}(0) = -\frac{3}{8 \cdot 2^{5/2}}$

→ so $P_3(x) = \sqrt{2} - \frac{1}{2\sqrt{2}}x - \frac{1}{8 \cdot 2^{3/2}}x^2$
$- \frac{1}{16 \cdot 2^{5/2}}x^3$

→ we get the approx to $f(1/2) = \sqrt{1.5}$ by evaluating $P_3(\frac{1}{2})$

→ <u>question</u>: how bad is the error, e.g., the difference between $\sqrt{1.5}$ and $P_3(1/2)$?

→ Compute

$$|R_3(x)| \leq \max_{c \leq |x|} |f^{(4)}(c)| \cdot \frac{|x|^4}{4!}.$$

we took $x = 1/2$

→ Note that

$$f^{(4)}(x) = -\frac{15}{16(2-x)^{7/2}}$$

Since we took $x = \frac{1}{2}$, we can take $c = \frac{1}{2}$ to make $f^{(4)}(c)$ as large as possible

So $$|f^{(4)}(\tfrac{1}{2})| = \frac{15}{16(3/2)^{7/2}}$$

$\rightarrow \left|R_3\left(\frac{1}{2}\right)\right| \leq \frac{15}{16\cdot(3/2)^{7/2}} \cdot \frac{\left(\frac{1}{2}\right)^4}{4!}$

which is pretty small.

Example 3:

Approximate

$$e^{0.2}$$

within an error of at most 0.01.

$\longrightarrow$ take $f(x) = e^x$, want to find an approximation to $f(\frac{1}{5})$ that is within $\frac{1}{100}$ of the exact value of the function.

$\longrightarrow$ first find the $N \geq 1$ we need to

approx. by $P_N(x)$ (expand about $x=a=0$)

$$\rightarrow \left|R_N\left(\tfrac{1}{5}\right)\right| \le \max_{c\le\frac{1}{5}} |e^c| \cdot \frac{\left(\frac{1}{5}\right)^{N+1}}{(N+1)!}$$

$$\le \frac{1}{100} \quad \text{(our requirement)}$$

- to get the largest value of e^c for $c \le \frac{1}{5}$, take $c=\frac{1}{5}$
- So find the smallest N that gives us

$$e^{1/5} \cdot \frac{\left(\frac{1}{5}\right)^{N+1}}{(N+1)!} \le \frac{1}{100}$$

- this gives $N=2$ by computation

→ find $P_2(x) = \sum_{k=0}^{2} \frac{f^{(k)}(0)}{k!} x^k$ when $x = 1/5$

$$f^{(0)}(x) = e^x \longrightarrow f^{(0)}(0) = 1$$

$$f^{(1)}(x) = e^x \longrightarrow f^{(1)}(0) = 1$$

$$f^{(2)}(x) = e^x \longrightarrow f^{(2)}(0) = 1$$

→ so we find that

$$e^{0.2} \approx P_2\left(\frac{1}{5}\right) = 1 + \frac{1}{5} + \frac{1}{50} \leftarrow \left(\frac{1}{5}\right)^2 \cdot \frac{1}{2!}$$

Taylor Series

A *Taylor series* is an infinite Taylor polynomial:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = \lim_{N \to \infty} P_N(x)$$

In other words, *a Taylor polynomial is the n^th partial sum of a Taylor series*.

If *a=0*, a Taylor series is called a *MacLaurin series*.

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

Common MacLaurin Series

(know/memorize these five series)

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, x \in \Re$$ ⟷ any real x

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, x \in \Re$$

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}, x \in \Re$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, |x| < 1$$ (this is the geometric series r=x)

$$\ln(1+x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}, |x| < 1$$ (we saw this last week)

Example 4.1:

Find a MacLaurin series for the following function:

$$f(x) = \frac{\sin(5x)}{x}$$

(Where does it converge?)

→ Start with the series for $\sin(5x)$:

$$x \cdot f(x) = \sin(5x) = \sum_{k=0}^{\infty} \frac{(-1)^k (5x)^{2k+1}}{(2k+1)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \cdot 5^{2k+1} \cdot x^{2k+1}}{(2k+1)!}, \quad (*)$$

this series converges for any real x.

$$x \cdot f(x) = (5x) - \frac{(5x)^3}{3!} + \frac{(5x)^5}{5!} - \cdots$$

(so division by zero is NOT a problem)

By (*), $f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \, 5^{2k+1} \, x^{2k}}{(2k+1)!}$,

and this converges for all real x as well.